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MASONRY ARCHES FOR RAILWAY PURPOSES.

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In dealing with the arch, the writer feels that it will be necessary to proceed carefully; as many Engineers have the impression that for any possible view of the subject some authority may be cited, and that all of them are still more or less at variance with the practical results obtained in construction. The writer of the present paper has no desire to propound any new theory of the arch, or to advocate any one in particular; but will rather make it his aim to compare the theories which have obtained the greatest currency with the object of ascertaining the conditions of construction and loading to which they are applicable, and also to indicate directions in which further investigation is still required to meet the case of railway arches carrying heavy rolling loads.

As it often happens, the discussion on this subject is largely due to misunderstandings which have arisen, owing to the different points of view taken by different authors, and the differing conditions which they have in their minds and do not sufficiently explain. The physical and mechanical properties which the materials are assumed to possess, the method of design and construction, and the purpose of the arch as regards the duty it has to fulfill, are seldom given with the clearness that such fundamental considerations should have. In most cases it is only by reading and re-reading, examining the conditions as expressed mathematically, and noting carefully some obscure hint occurring in the course of the investigation, or even by reference to the examples cited, that the assumed conditions can be inferred. The investigation in different countries has also been carried on without sufficient reference to what has been ascertained elsewhere.

The older theories of the arch need not be reviewed here. The modern study of the subject dates from the introduction of railways, and originated with Méry and Moseley, who were the first to investigate the properties of the curve of pressure. (1) The various theories on the subject have now resolved themselves into two leading ones, which under more or less modified forms have obtained by far the greatest currency. The first of these may be briefly stated as follows: the curve of pressure must remain within the middle third of the arch ring, in order that the stability of the structure be assured. The second is based upon the principle of least resistance, and maintains that the thrust developed at the key will not exceed the least amount capable of maintaining equilibrium. The first of these has great currency amongst English authors, and being endorsed by Rankine, is often referred to as an axiom. On the Continent the second is largely adopted as an improvement on older theories, and prominent among its exponents are Dupuit and Scheffler. Since Graphical Statics has developed so widely from the foundations laid by Culmann and Lévy, (2) it has been almost universally adopted as the best method for the investigation of the stability of the arch. It must be noted, however, that so far as the arch is concerned it is only a method, and does not afford a solution apart from more general principles. We have also the help of a large amount of observation and experiment, but these must be distinguished from each other. In observing structures during and after their erection all the conditions are present, but the difficulties of observation admonish corresponding care in the inferences drawn. In making experiments with models, these difficulties can be largely removed, but the material employed being usually different, the difference in physical properties must be allowed for.

A complete theory of the arch must apply to the finished structure fulfilling the purposes of its construction; but before proceeding to examine the theories referred to, we may first mention the conditions which accord with the ordinary principles of statics and for which no special theory is required; and we must also examine carefully the physical and mechanical properties of the material with which we have to deal, as explained in works on the subject. The design, method of construction, and purpose of the arch with respect to the loading it has to carry, furnish further conditions which must be taken up in connection with the theories themselves.

There are only two ways in which a perfect arch can be constructed, and these are the converse of each other; either the curve of the arch must be designed to correspond with the loading, or the loading must be placed to correspond with the curve adopted. This can always be done when only dead weight or a stationary load are in question. For example, in the case of uniform loading the catenary and the parabola are familiar as the forms required, according as the load is uniformly distributed along the curve itself or along a horizontal line; and a circular arch may also become equilibrated by a suitable distribution of the loading upon it. If these methods were adopted, the curve of pressure would coincide with the centre of the arch ring throughout, and the direction of the pressure would be perpendicular to the joints. There would be entire harmony between principle and practice, and the only condition of stability would be the absolute pressure admissible in accordance with the material used. When it is the loading that is prescribed, the form required for the arch is usually difficult of construction, owing to the continual change in the radius of curvature; but notwithstanding this, close approximations to the true forms have been adopted with economy on aqueducts and canals. Their advantages disappear however when a change in the position of the loading has to be allowed for; and for railway purposes semi-circular, elliptic and segmental arches are substituted. Although this is the case, a valuable improvement in the stability of the arch may be obtained by a suitable distribution of the dead weight. To simplify construction the ellipse is often replaced by circular arcs, and there are very elegant methods of describing these by means of a series of centres, (3) to which the name of "basket-handle arches" has been given.

When both the form of the arch and the loading are defined, the arch is subjected to conditions diverging from those required for perfect equilibrium. This is unfortunately the case we are called upon to consider in practice, and for which a theory has to be found. The curve of pressure no longer cuts the joints of the arch ring at the centre, and a weak point develops at the haunch known as the joint of rupture where the curve passes nearest to the intrados.

In considering a series of voussoirs in such an arch, we must first determine how the pressure on the surface of a joint will be distributed when the resultant of the external forces is known. We will suppose the resultant known in position and magnitude, and for simplicity we will take it perpendicular to the surface of the joint. We further take the surfaces at the joint to be in perfect contact, but without cohesion, so that only compression can be resisted and not tension. With regard to stone and brick we have also some evidence that we are dealing with a material that is elastic and therefore also compressible. As boys we learn that a marble will bounce on a paving stone; the amount that a brick chimney will oscillate in the wind cannot be attributed to the mortar alone; and we have further the corroborative evidence that stone expands with heat. (4) Co-efficients of elasticity have been determined for glass and also for slate; but not for the materials ordinarily used in construction and when under direct compression, so far as the writer is aware. It is the existence of elasticity and not its amount however, that affects the question; and it is doubtful whether any solids exist which are either absolutely incompressible or infinitely hard. Without going further into those physical properties of stone and brick for which the experimental data are so meagre, it is enough that we have the right to infer that they are elastic, and to apply to masonry the same Theory of Elasticity as to other materials. In accordance with this theory, the particles which are in a plane when the body has its natural shape will remain in a plane when the body becomes deformed under compression, the new plane being either parallel or inclined to its original position. Apart from this theory we have no means of determining the distribution of pressure at a joint; and although something might be inferred from the final conditions obtaining on the failure of the arch, we could have no knowledge of the internal strains at any other time. We find accordingly that all authors describe the pressure at the joints as being distributed in a lineal ratio to be determined by the position of the resultant; and in doing so, whether they give the explanation or not, they are assuming either that stone and brick are elastic or that they act as if they were; and not only so, but that they conform to the accepted Theory of Elasticity. (5)

With regard to the nature of the contact at the joint, the most accurate workmanship could not be depended on to make it perfect without the intervention of mortar. Nearly all authors agree in considering this the only function of the mortar, (6) and even in the case of cement the adhesion is not counted upon, but is left to form part of the margin of safety. Some few make an exception of brickwork built in cement, and consider both tension and compression as occurring at the joints. This amounts to transferring brickwork from the conditions of masonry to those of a metallic arch, and limiting masonry to stonework only; but there are few who make such a distinction between stone and brick. The only remaining action considered at the joint is friction, which renders an inclination of the resultant compatible with stability when within the limits of the angle of friction.

It follows then directly from the theory of elasticity that when the resultant passes through the centre of the joint, the pressure is uni-

formly distributed over its surface; and also that as it moves from the centre the pressure becomes unequal, till on reaching a point at one-third from the edge the pressure becomes zero at the other edge, as shown in Fig. 1. This is necessarily the limiting position of the resultant for which the whole joint can be in compression; and the theory that the curve of pressure must be within the middle third of the arch ring is therefore equivalent to maintaining that every joint in the arch must be in compression over its entire surface. This corresponds with a much higher degree of stability than many arches actually have; and we may therefore follow the effects produced at the joint as the resultant passes out of the middle third and approaches the intrados. The joint will no longer be entirely in compression, but the pressure will extend over a width equal to three times the distance of the resultant from the inner edge, and the pressure at that edge will be double the mean pressure on that area. As to the remainder of the joint it carries nothing, not being capable of bearing tension. (Fig. 2.) Strictly speaking the outer edge will open but only by an amount that is entirely inappreciable. It is capable of calculation, being proportional to the compression of the stone at the inner edge; but the compressibility of stone is so infinitesimal that practically it cannot be measured; and the "opening" of the joint under such circumstances is therefore entirely theoretical. This position of the resultant is then still compatible with stability. As the resultant continues to approach the edge, the same action continues in an increasing ratio, until the pressure at the inner edge reaches the ultimate strength of the material, when crushing will take place, rotation through a small angle will ensue, and the joint will open visibly at the opposite edge. This is the true limiting position of the resultant in actual cases; and for hard and sound material it may be very near the intrados. That the resultant may be very near the edge of the joint, without causing the failure of the arch, is also held as proved by authors who base their conclusions on experiment. Prof. Cain shows by experiments with wooden models that at the joint of rupture the curve of pressure passed on an average at only one-eighteenth of the width of the joint from the edge without rupture ensuing. He also calculates that in the bridge of Neuilly the thrust at the crown might pass within less than two inches from the edge without crushing the material. (7) We must therefore conclude that in the case of actual masonry, the curve of pressure may approach extremely near either to the intrados or the extrados without rupture occurring in the arch. This shows the great difficulty of deducing a theory from existing structures, as so little can be inferred from the bare fact that a structure remains standing.

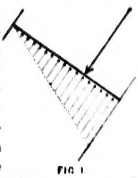

FIG 1

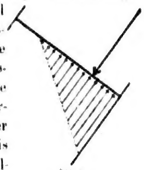

FIG 2

Let us now take the accompanying outline to represent a portion of the arch-ring in a full arch, either semi-circular or elliptic. We will take it in the usual way to represent by its area the weight of a unit of width in the direction of the axis; and the loading may also be shown as an area representing its average amount per unit of width. We will also consider the loading as placed symmetrically on each side of the key, and a half-arch will therefore be sufficient for the figure. For the forms of loading we have ordinarily to consider it may be broadly stated that the curve of pressure will resemble a parabola. It is easier to determine its general form than to determine its position in the arch ring; but if such a curve is placed within an arch ring of either circular or elliptic from, it will approach the intrados at to opposite points at the haunches, and at those points it will be parallel to the tangent at the intrados. This determines the positions of the joints of rupture, one of which is shown at CD; and if we also take a vertical joint AB at the key, these are the only ones which the curve of pressure will necessarily cut at right angles. The moments of the forces must therefore be taken with reference to the portion $ABDC$; as to divide the arch ring at any other point would introduce a radial force representing friction.

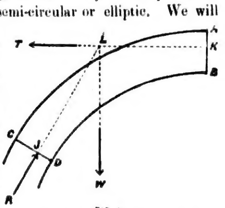

FIG 3

We have then three forces to maintain equilibrium. The thrust T is horizontal on account of the symmetry. The line LW is either a vertical through the centre of gravity of the area representing the arch ring itself between the joint of rupture and the key; or it is the resultant of the weight of the arch ring, and the pressures upon its extrados between the same points. When these pressures are all vertical, LW will also be vertical and may be termed the *Gravity line*. When the joint of rupture is known in position, it becomes known in position direction and amount; and the resistance R also becomes known in direction. Thus of the three forces two are known in direction only; but to maintain equilibrium they must intersect in a point at L, and this may be termed the *Condition of intersection*. The position of the joint of rupture must correspond with this condition.

It is evident that if the position of the points K and J can be determined, the curve of pressure also becomes determinate,* and consequently the value of the thrust T. These are the ***Determining points*** in the curve, but unfortunately we have no means of deciding their position *a priori*. The reason for this is only too apparent. We have an external moment produced by the load; but no equivalent moment of resistance is developed in an arch ring as it would be in a metal rib. Hence until the points of application K and J can be determined, the equation between the moments of T and W remains indeterminate, and we cannot therefore proceed either to an analytical or a graphical solution. This abundantly accounts for the great variety of opinions amongst authors. All the theories of the arch can be classified in accordance with the positions they attribute to these points. In mathematical works it is often very difficult to discover the reasons for the choice made; and in works treating the subject graphically, while there is ample explanation of the method of drawing the curve of pressure, there is usually a lack of clearness as to the principles upon which the position of these determining points depends. Although an author may give good reasons for their position in one case, he may be too hasty in making a general application of the result to all cases. It is for this reason that some theories have fallen into discredit, although they undoubtedly have some good features in them, and may be quite applicable under certain conditions.

The direct determination of the position of these points has been attempted at about the same time by Scheffler and Dupuit, based upon the principle of least resistance, but applied in very different ways. This principle is due to Moseley (8) and may be briefly stated thus: the thrust developed at the key will not be greater than the amount which is just sufficient to maintain equilibrium. As it is evident that the least amount of thrust will correspond with the co-incidence of K with A, and J with D, Moseley takes the curve of pressure as tangent to the extrados and intrados at the key and joint of rupture respectively. Scheffler (9) generalizes the same view, and takes a curve of pressure tangent to the extrados and intrados of the arch ring for all cases of loading, whether symmetrical or not. Such a curve corresponds obviously to the case of a structure in which both the arch ring and abutments are everywhere on the point of giving way. If the abutments are about to yield, and the joints CD and AB are about to open, we have the required conditions, and failure will take place in the way so often represented in text-books. Although it can be shown that the curve will only take this position in extreme cases, it may nevertheless be useful as a means of investigation. This theory is also in favor with experimentalists, (10) as it accords with the final conditions which can be observed at the moment of failure. An arch might be designed in accordance with it, by giving it minimum dimensions throughout, and afterwards adding material to cover the possibly greater values of the thrust and other pressures that might occur.

Dupuit (11) applies the same principle but proceeds more carefully. He brings it into relation with the ordinary process employed in the building of arches, and the effects that can be observed when the centres are struck. By following the steps he takes, we will be best able to judge of the principle, and to determine the range of its applicability to the arch.

The following summary of his reasoning is taken from Claudel: (12) While the arch remains on its centres the pressure in the arch ring is zero throughout; and as the centres are eased, the pressures found in the finished arch are developed. These pressures therefore must pass through all the intermediate values from zero upward; and in accordance with the principle of least resistance, he infers that when once the least amounts necessary for stability are reached, there is no further reason why the pressures should continue to increase. In the movement which occurs when the centres are struck, there is something fixed and certain which depends only upon the curve of the arch, and also an amount of uncertainty depending upon the nature of the materials and the methods of construction, so that the uncertainty finally remaining is reduced to such narrow limits as to have no further interest from a practical point of view. While the arch remains on its centres the curve of equilibrium is entirely beneath the arch, and is determined by the intersections of the verticals through the successive centres of gravity, with the production of the joint lines. (Fig. 4.) This may be called the curve of static equilibrium, and corresponds with an absence of pressure in the arch ring. This curve serves merely to indicate the point in the arch around which rotation would tend to take place. As a small thrust develops at the key, this curve rises toward the arch and indicates more distinctly the position of the point of rotation D. When the thrust increases sufficiently to make the curve of pressure pass through the point D, it is evident that it is sufficient to maintain equilibrium. There is therefore no reason why the curve should pass this

*As the curve of pressure is symmetrical for symmetrical loading, and the points K and J imply direction as well as position, they are mathematically equivalent in the complete arch to six determining points on the curve. It would therefore have to be of a very high order to take two different positions between K and J.

point and take up a position in the interior of the arch, as such a position would correspond with a greater amount of thrust at the key. The point D being determined we know in what direction the arch will move if the half arch is for a moment left unsupported. The path which each point in the arch tends to take is evidently the arc of a circle around D as a centre. Thus the points A and B will tend to describe the small arcs AA' and BB' corresponding to the same angle at the centre D. The resistance of the other half of the arch however prevents actual motion; but it follows that the points A and B support unequal pressures, as the amount of compression at these points is evidently proportional to the horizontal distances through which they would move respectively, if they were free to do so. The effect produced by the other half of the arch is to maintain the joint AB in a vertical position; and it follows that the horizontal compression at each point is proportional to the decrease in length of the horizontal projections of the radii drawn from D to A and B respectively. It is easily shown that this proportion is the same as the ratio of $h+t$ to h. If we take (Fig. 5.) two lines AP and BQ in this ratio, it follows from the accepted theory of elasticity that the resultant of the whole pressure on AB will pass through the centre of gravity of the trapezium $APQB$. The point K is therefore not at the centre of AB, but always a little above it. For the limiting cases, if $h=o$ as in the *plate bande* or straight arch, $BK=\frac{2}{3}t$; and if $h=$infinity, $BK=\frac{1}{2}t$. This reasoning assumes that the point D is rigidly fixed, or in other words that the abutments are immovable and incompressible; and here we meet with the uncertainty mentioned at the outset. If the point D were to yield laterally, the point K would rise in proportion; and would approach as near to A as the strength of the material allowed.

The figure employed in the demonstration is for a segmental arch; but for a full arch it can be shown by similar considerations that the curve of pressure must be tangent to the intrados where it meets it, and that the point of contact determines the position of the joint of rupture. In the actual arch the curve must be sufficiently within the intrados at the joint of rupture to prevent crushing. He further shows that the positions which the joint of rupture can take are always between one half and two-thirds of the rise, in the case both of the semi-circle and ellipse. The exact position depends on the ratio of the thickness of the arch ring to the rise of the arch, as this affects the distance of the centre of gravity from the key; but for the ordinary proportions found in practice, the joint will be very nearly at half the rise.

Dupuit's conclusion is then that with unyielding abutments the position of the curve of pressure will be determined as follows:—

The point K will be at the centre of $A\ B$ for a semi-circular or elliptical arch, and at the upper third of $A\ B$ for a plate-bande. For intermediate forms it will lie proportionately between these limits.

For semi-circular and elliptical arches the curve of pressure will be tangent to the intrados at the joint of rupture, and the joint of rupture is to be determined by this condition. For segmental arches of less amplitude than the arc between the joints of rupture in the full arch, the curve of pressure will pass through the point D at the springing.

In examining this theory it is evident that the determination of the point K would not be appreciably affected if the point J were to move out along the joint of rupture; but on the other hand it does depend entirely on the supposition that the abutments are unyielding. If the abutments give laterally under the effect of the thrust, the joints of rupture will open and also the joint at the key. The curve of pressure will then necessarily pass through A and D; but the value of the thrust will be considerably less than for a curve passing through K and D. We see then that any yielding of the abutments gives an immediate reduction in the value of the trust; and therefore the abutments require greater strength to resist yielding than to stand after a slight motion has taken place. This accounts for the regained stability of structures in which a slight movement has occurred. We have also in this a distinguishing difference between the theories of Dupuit and Scheffler. As examples of cases in which the condition of unyielding abutments becomes perfectly realized, we may mention a segmental arch springing from the solid rock, or a series of arches all equal and equally loaded; but it will be practically fulfilled in any case in which the abutments themselves and their foundations are sufficiently resisting.

It cannot be too carefully noted, however, that Dupuit's reasoning is based upon the consideration of the arch ring itself while standing alone. For that case then we must consider his conclusions as established. But in supposing that the curve of pressure will remain tangent to the intrados when the arch ring carries the weight of the spandrel walls, backing and filling, Dupuit is certainly going beyond the limits of what he has actually proved; although he renders most valuable assistance to the consideration of the subject by indicating the true starting point and the first steps to be taken before more complicated conditions can be examined with advantage. His method for the completed arch is to find by trial a new position for the joint of rupture which will fulfill the condition of intersection, the tangent at the joint and the thrust at the key having to intersect on the gravity

line. This new position will be lower than for the arch ring itself, as the gravity line will be further from the key.

It is not usual, however, to complete the whole structure before removing the centres, as any settlement of the arch ring would render very uncertain the actual distribution of the pressure upon it. To suppose the arch built in this way in order to apply the principle of least resistance to the finished structure, is to introduce a large and unnecessary amount of uncertainty. Now when the gravity line moves further from the key it is evident from the condition of intersection, either that the joint of rupture will take a lower position, or that the point J will move out from the intrados to maintain $L\,J$ at right angles to $C\,D$. To decide between these alternatives we must first examine the question as to how the pressure from the spandrel is transmitted to the back of the arch ring, as it is possible that when the arch ring carries its weight the line $L\,W$ may no longer be vertical. As a limiting direction, the pressure may be everywhere at right angles to the extrados; but this condition can only be realized by taking exceptional precautions with the object of obtaining the "hydrostatic arch," and we cannot therefore suppose it to be the usual case. Again, the pressure may act along inclined lines similar to those given by the pressure of earth. This will be the case for deeply buried culverts and for sewers; but these we cannot now consider, as their stability depends on conditions too far removed from those we have before us. Some Engineers maintain that the pressure of masonry will also act similarly along inclined lines of fracture. It is possible that this may be the case to some extent, especially towards the haunches; but for the central portion of the arch, it is sufficient to adopt the ordinary supposition that each voussoir carries the portion of the load vertically above it. As regards the haunches it is an almost invariable rule to carry up the backing to the level of the joint of rupture as found for the arch ring, before the centres are struck. When this is done the question is much simplified, as the whole of the masonry below that joint can then be taken as forming in reality a part of the abutment. We will assume then that it is only the remainder of the spandrel which is added after the centering is removed; and this has the sanction of the best constructors. When the arch is built in this way, we are not at liberty to suppose that the joint of rupture will take up a lower position than it had in the arch ring, and we are left to the alternative that on the addition of the spandrels the point J moves out from the intrados. We see at the same time the advantage derived from adopting this method of construction.

This also seems the more probable when we compare the form of the spandrel with a form of loading which will make the curve of pressure coincide with the centre line of the arch ring throughout. This form can readily be found by reversing the ordinary graphical process. The centre line of the arch ring is assumed to be the curve of pressure, successive tangents to it are drawn, and a force diagram constructed in which the successive parts represent the weights that have to be applied along the curve. (See Greene's "Arches," Art. 140.) By plotting these parts from the intrados as ordinates, the weight of the arch ring is taken into account, and the area indicated by the ordinates is the loading required. Its form is given in Fig. 6; and on comparing this form as far as the joint of rupture with the usual form of the area representing the weight of the spandrel, the similarity is sufficient to enable us to infer that the addition of the spandrel improves the position of the curve of pressure. This accords with experiment, and with observations on completed structures. It would justify also the practice of increasing the thickness of the arch ring toward the springing; for while this would add nothing to the strength of the arch ring standing alone, it becomes of real service when the structure is completed. There is therefore every reason in favor of the conclusion that when on the addition of the spandrel the gravity line takes up a position further from the key, the joint of rupture will not change its position, but the point J will move out from the intrados. This change however, will not affect the considerations which led to the determination of the position of K.

For the structure completed as above described, and with unyielding abutments, the following general method of finding the determining points is suggested as harmonizing with the best discussion of the subject as given by different authors:—

The point K will remain at the same position as in the arch ring.

For a semi-circular or elliptical arch the joint of rupture will remain at the same position as in the arch ring, and the point J will be found by drawing from L where T and W intersect, the line LJ at right angles to CD. Similarly for a segmental arch, the point J will be determined by drawing LJ parallel to its former position in the arch ring.

To make the position of the points K and J dependent upon the position of the gravity line as this method does, is much more reasonable than to give them the same arbitrary position for all cases. It also affords a means of ascertaining the most advantageous distribution of weight in the spandrel. As a rule the point J will be only a short distance out from the intrados; but in extreme cases, if there is a large amount of loading over the haunches, the gravity line may be so far

from the key as to bring the point J to the centre of CD and eventually even to the extrados. We have been assuming, however, that the loading acts vertically; but in such a case the assumption may reach a limit beyond which it is no longer true. When the loading is so highly concentrated over the haunches it is more than probable that the direction of the pressure becomes inclined to the back of the arch ring. This is difficult to estimate in amount, but in semi-circular or elliptical arches it will be quite sufficiently allowed for by supposing that when J reaches the centre of CD, the line LJ then becomes tangent to the centre line of the arch ring. In a segmental arch in like manner, the pressure of the spandrel will prevent J from passing the center of CD. It would, however, be preferable to increase the thickness of the arch ring towards the springing, or to re distribute the weight in the spandrel, rather than to count upon any inclination in the pressure above the joint of rupture to improve the position of the curve of pressure. Such positions of J are not likely to occur for any ordinary form of spandrel.

Let us now compare this method with the theory that the curve of pressure must remain within the middle third of the arch ring. This theory is endorsed by Rankine, and the weight of his authority has tended more than it should to deter further investigation. His statement is that the stability of an arch is secure if the curve of pressure can be drawn within the middle third; and he goes so far as to say that although arches have stood and still stand in which the curve lies beyond the middle third, the stability of such arches is either now precarious or must have been precarious while the mortar was fresh. (13) We must first endeavor to ascertain the nature of the arch and the conditions of which Rankine is treating. It is to be inferred from the cases he takes up and the examples he gives, that he is considering arches for which the amount of the moving load can be neglected in comparison with the weight of the structure itself. Prof. Wm. Allan in his "Theory of Arches," which he describes as being an amplification and explanation of Rankine's chapters on the subject, takes for granted that this is Rankine's point of view: "In all stone or brick arches the changes in the curve of pressure due to passing loads are usually slight, because the weight of such passing loads is generally small compared with the weight of the arch itself and its backing." (14) This statement affords the key to Rankine's explanation of the subject. If this had been distinctly pointed out in his works, a large amount of discussion and misunderstanding might have been avoided.

The same explanation of the range of application of the theory of the middle third is given in the article on "Bridges" in the last edition of the *Encyclopædia Britannica*: "The masonry arch differs from the superstructure of other bridges in the following respect: it depends for its stability on the presence of a permanent load specially arranged, and so considerable in amount that the changes produced in the direction and magnitude of the stresses by the passing load are insignificant." (15)

The theory of the middle third corresponds then to the case of the finished structure without appreciable moving load. To find in accordance with the method suggested above the limits within which the gravity line must keep in order that the curve should remain within the middle third of the arch ring, is merely a matter of geometrical construction; and we have thus a ready means of comparison by which to verify the results. Take for example the case of a semi-circular arch of radius r, in which the arch ring has a uniform thickness t, and the joint of rupture is at 30°. The possible limiting positions of K and J for a curve remaining in the middle third will be, the upper third at the key and the lower third at the joint of rupture, or the lower third at the key and the upper third at the joint of rupture. The distance y of the gravity line from the key corresponding to these limits will be: $y = r \tan 30°$ and $y = (r + t) \tan 30°$ Or numerically, for semi-circular arches of 10 ft. and 100 ft. diameter, in which the thickness t is determined by Rankine's formula, the limits will be :—

10 ft. arch 0.577 r and 0.664 r. 100 ft. arch, 0.577 r and 0.605 r.

As it happens, the gravity line for spandrels as generally built has very nearly this position in full arches and a corresponding one in segmental arches; and this accounts for the currency of the theory as applied to the conditions supposed.

This theory also implies, as we have seen, that every joint in the arch must be entirely in compression. This corresponds with a much higher degree of stability than is necessary under quiescent loads; but the advocates of the theory take it practically as a margin of safety. They consider that if the curve is within the middle third in the structure bearing its own weight, it will be sufficiently stable under any moving loads it may be called upon to carry. (16) As the theory is fairly in accord with the form which the spandrel usually has, it may be sufficiently near the truth as regards road bridges; but it is taking quite too much for granted in the case of arches carrying heavy engine loads. It could only be brought into reasonable agreement with such cases by lowering the springing and increasing the depth and weight of the spandrel, till the moving load became relatively small enough to neglect; but such a construction would be inconsistent with economy. To have the curve of pressure within the middle third of the arch ring is very desirable and should be aimed at; but it cannot be laid down as a

principle that it must be so. It is in reality only a special case coming under a more general method; and it would therefore be more correct to work in the converse direction, and to find the depth of the spandrel and the amount of backing required to make the curve take the desired form, as nearly as the case will allow.

We may now proceed to consider the arch carrying a live load symmetrically placed. This load we may reduce in the usual way to an area representing an equivalent amount of masonry.

The load on Railway arches is now often as great or greater than the weight of the structure itself, which shows again how unsafe it is to apply methods which depend upon its amount being neglected. If the load is uniformly distributed, the centre of gravity of the whole area representing arch ring spandrel and load between the joint of rupture and the key, will almost always be further from the key than the centre of gravity of the arch ring itself; and this being the case, the reasoning given before will be entirely applicable. The joint of rupture will still have the same position as in the arch ring, and the general method as given for the full arch and the segmental arch will be the same.

The gravity line with a distributed load will be nearer the key than in the unloaded arch, and the point J will also be nearer the intrados. With an excessive distributed load, or with a load increasing towards the key (while still symmetrical as regards the two sides of the arch), the point J may reach the intrados. This is an unfavorable position, but it does not necessarily compromise the stability of the arch. It is desirable if possible to prevent its occurrence, either by thickening the arch ring, modifying the form of the arch, or re-arranging the weight in the spandrels in order to keep the gravity line further from the key.

If for any positions of the line load which the arch has to carry the gravity line is brought still nearer the key, the line $L\ J$ will continue tangent to the intrados and the joint of rupture will rise. In such extreme cases it is possible that the point K may also rise at the key, even with unyielding abutments. It appears both from experiments with models, and by comparison with the form of the funicular polygon, that the curve of pressure will rise immediately under any highly concentrated load; and with a moving load such concentration is necessarily greatest at the crown. It would therefore seem probable that the thickness of the arch ring at the crown depends more directly upon the heaviest concentrated load passing over the arch than on any other consideration. Vibration on the contrary has more effect towards the haunches.

We are still dependent, however, upon empyrical formulae for the thickness at the crown. Early authors suggested an equation of a linear form; but this has now been replaced by the forms

$$t = C\sqrt{s} \quad \text{or} \quad t = C\sqrt{r}$$

t being the thickness required, s the span of the arch, and r the radius at the crown, all in feet; and C a constant. (17) The values most generally adopted for C are those given by Rankine and Dupuit, which are as follows:—

Rankine. $t = \sqrt{0{\cdot}12\ r}$ for a single arch.

$t = \sqrt{0{\cdot}17\ r}$ for an arch of a series.

Dupuits. (Co-efficients reduced for feet)

$t = 0.36\ \sqrt{s}$ for a full arch.

$t = 0.27\ \sqrt{s}$ for a segmental arch.

Rankine's introduction of the radius of curvature at the crown instead of the span, is based upon a comparison between the arch and an elastic rib; but he deduces the co-efficients from actual examples. With regard to the greater value in the case of arches in series, he considers that yielding is more likely to take place when a loaded arch stands between two unloaded ones, than if it stood between abutments. Dupuit's formulae are based directly on very numerous examples, a large proportion being Railway bridges. He refers only to actual construction as justifying the lower co-efficient he gives for segmental arches. (18)

The starting point for all such formulae is the semi-circular arch of diameter d; and for it a comparison between the formulae can readily be made. We have for a single arch:—

Rankine. $t = 0{\cdot}24\sqrt{d}$ Dupuit. $t = 0{\cdot}36\sqrt{d}$

The greater value given by Dupuit's formula is probably due to the greater proportion of Railway arches on which it is based, and perhaps also to the lower average strength of the stone used in France. If we consider the span to remain the same, and the arch to change from a semi-circle to a straight arch by passing through the intermediate segmental forms, we find that Dupuit would give the arch ring a diminished thickness. Rankine's formula, on the contrary, would make it continually increase, and on arriving at the straight arch the thickness would be infinite, or in other words a straight arch would be theoretically impossible. This may result from the supposition that the joint lines are always radial; and his formula might still be applied to the straight arch by taking the point from which the joints radiate as the supposed centre. An intermediate formula has been proposed by Trautwine in the last edition of his Pocket-book:—

$$t = 0.25\sqrt{r + \tfrac{1}{2}s} + 0.20 \text{ (for feet)}$$

For a semi-circular arch this is practically the same as Rankine's, with the difference of the constant added. It does not increase the thickness so rapidly as his, but gives also an infinite thickness to the st aight arch.

There is still need of agreement respecting the rational basis for such formulæ, as they differ in principle. They could only be satisfactorily compared by dividing the examples on which they are based into classes, in accordance with their purpose and the conditions of their construction.

To determine the stability of the abutments it is only necessary to continue to the foundations the curves of pressure corresponding to the various cases considered. As we have already had occasion to notice, the abutments require greater strength to resist yielding than to stand after a slight motion has taken place; because the position of the point *K* as determined for unyielding abutments, corresponds to a greater thrust and greater pressures throughout the structure than if it rose to *A* through the opening of joints in the arch ring. With symmetrical loads the greatest resistance will be required when the arch is completely loaded; and although the effect of partial loading may possibly be greater, this is almost invariably left to the margin of safety. If the curve of pressure strikes outside of the middle third of the base, there is a portion of the abutment whose weight should not be taken into account. (Fig. 7.) The effective width of each course is then only three times the distance from the curve to the back of the abutment; and the curve of pressure should be drawn again, omitting the weight of the remainder. This precaution is especially necessary in the case of segmental arches. When all the cases of loading are considered, the abutment will be undoubtedly stable when the resulting curve is sufficiently within the base to prevent excessive pressure, provided the foundations are thoroughly sound. It will be quite unnecessary to double the thrust from the arch as recommended by some authors, unless indeed this is done to allow for positions of the load which are not considered.

When arches are built in a series, somewhat different conditions arise. When the weight only of the structure is to be taken into account, the pressure on the piers is vertical and the resistance to crushing is the only consideration. For arches carrying a moving load, a pier is the most unfavorably situated when it is between a loaded and an unloaded arch. The reaction at the joints of rupture in the two arches respectively when compounded with the weight of the pier, give the final resultant, which in a pier should always meet the base within the middle third. Before failure can take place however, the thrust of the loaded arch will develop a much greater reaction at the joint of rupture of the unloaded arch than the amount due to its weight only. If the loading increases to the limiting amount, the loaded arch will fail in the ordinary way, but the joints in the unloaded arch will open in the reverse direction, corresponding to a curve of pressure passing through *B* and *C*, and therefore also to a largely increased thrust. Such a thrust should not be counted upon except in the case of small arches used as counterforts, or transverse arches in the interior of an abutment; but it will at least be allowable to consider the pressure at the joint of rupture of the unloaded arch to become equally distributed under the influence of the thrust from the adjoining arch, and so to place *J* at the center of *C' D* in the unloaded arch.

The various theories have now been compared in the endeavor to point out the conditions under which they are applicable, and to ascertain the position of the determining points in the curve of pressure which best accords with them in the various cases considered. It will be unnecessary to proceed further into detail, as the method of drawing the curve, when these points are known, is fully given in works on Graphical Statics. In applying the graphical method, a distinction has in some cases to be made between the "curve of pressure" and the "line of resistance," according to the division of the arch ring into vertical laminae or actual voussoirs. The method of passing from one to the other is given by Prof. Clarke. (19) The distinction between these curves was originally pointed out by Moseley, and is well illustrated by Dubois. (20)

We have not considered the case of unsymmetrical loading, as so few authors touch upon it at all, and those who do differ so widely in opinion. When an arch is loaded symmetrically, it has been maintained that the thrust at the key will remain the same if the load is removed from one half, leaving the load on the other side only. This seems plausible at first sight, as this thrust is undoubtedly sufficient to support the loaded side of the arch; but on continuing the curve on the unloaded side it will often pass entirely out at the extrados in existing structures which could not possibly stand if this were the case. On the other hand, a mean between the values of the thrust for the loaded and unloaded arch is not sufficient to support the loaded side. From a general theorem given by Collignon (21), it would appear that with unsymmetrical loading friction is developed at the key, or in other words the direction of the thrust is inclined towards the less loaded side. This accords also with the direction of the tangent at the centre of an unequally loaded catenary. In the case of an arch with an engine load on one side only, an inclination of 5° to 10° in the line of

thrust will usually prove sufficient to keep the curve within the arch ring; and this is a very moderate amount compared with the limiting angle of friction. The curve corresponding to such an inclined thrust is found to rise under the load, as would be expected. The joint of rupture tends to fall on the loaded side and to rise on the unloaded side; and if the arch is so constructed that the joint of rupture cannot take a lower position, the value of the thrust for any assumed inclination can be found after a few trials by equating the moments on the loaded side only. So far as the writer has been able to determine, the effects of such partial loading may be actually greater than with a complete load, especially as regards the abutments. Although we may infer the general opinion to be that the results would not differ largely from those already obtained for complete loading, the effects produced should not be left to the margin of safety in the structure; and this part of the subject therefore requires further investigation.

REFERENCES.

(1.) *Curve of Pressure.* Its properties were first discussed by Méry in 1827, though his work remained in manuscript till 1840. In the meantime Moseley's article appeared in 1833. (See No. 8.)

(2.) Culmann. "Die graphische Statik," Zurich, 1866.

Lévy. "La Statique graphique," Paris, 1874.

(3.) *Basket-handle arch or false ellipse.* A summary of various methods proposed for drawing these is given by Morandière, "Traite de la Construction des Ponts et Viaducs," chap. 3, pages 168 to 181; Paris, 1874. For the methods proposed by Michal and Perronet, see also Claudel, "Aide-mémoire des Ingénieurs," Art. 854. 9th Edition; Paris, 1877.

(4.) *Expansion of stone by heat.* Mentioned by Stoney as occurring in masonry arches; "Theory of Strains in Girders," &c., Ch. 19, Art. 414. Observed in an experimental arch of 148 feet span designed by Romany for the Pont du Louvre, Paris; "Annales des Ponts et Chaussées," 1866, 2e semestre, page 10. Also cited in Morandière's "Traité," page 217.

(5.) *Theory of Elasticity applied to Masonry.* Well explained by Collignon, "Résistance des Matériaux," Arts. 30 to 66, and Art. 228.

(6.) *The function of mortar.* See Moseley, "Engineering and Architecture," 2nd Edition, page 482. London, 1855.

(7.) *Limiting positions of the Curve of Pressure.* See pages 85 and 28 in "A practical Theory of Voussoir Arches," by Prof. Wm. Cain; Van Nostrand's Science Series, No. 12, 1874.

Principle of Least Resistance.

(8.) Originally published by Moseley in the "Philosophical Magazine" for October, 1833. See also his "Engineering and Architecture," Art. 332. The priority claimed for Coulomb cannot be established.

(9.) "Theorie der Gewolbe und Futternzauern," Scheffler, 1857. Translated into French by Victor Fournié, "Théorie des Voûtes," Paris, 1864.

(10.) Scheffler's theory compared with experiment, "Voussoir Arches," by Prof. Wm. Cain; Van Nostrand's Science Series No. 42, 1879.

(11.) Dupuit's Theory. Published originally in the "Annales des Ponts et Chaussées," 1858. In a separate form as "Traité de l'équilibre des Voûtes;" Text and Plates. Paris. 1870.

(12.) Summary of Dupuit's work given in Claudel, "Aide-mémoire des Ingénieurs," sections 870 to 934. 9th Edition.

Theory of the Middle Third.

(13.) Rankine, "Civil Engineering;" Arts. 123 to 141, and 276 to 298.

(14.) Rankine's chapters on the Arch amplified and explained by Prof. Wm. Allan, "Theory of Arches," Van Nostrand's Science Series No. 11, 1874.

(15.) "Encyclopædia Britannica," ninth Edition, 1876. Article "Bridges," by Prof. Fleeming Jenkin.

(16.) See foot note to Art. 178 in "Graphical Statics," by Prof. Dubois New York, 1883.

Thickness of the arch ring at the key.

(17.) The equation of the form $t = C\sqrt{r}$ was first proposed by J. T. Hurst, "Building News," Feb. 27, 1857; though the corresponding form was adopted independently on the Continent.

(18.) For the bridges on which these formulæ are based, see Rankine's "Civil Engineering," Art. 290; and Dupuit's "Traité," Chapter 7 and Plate 5.

(19.) *Graphical methods.* Change from vertical laminæ to real joints, given by Prof. Clarke, "Graphic Statics," Art. 52. London, 1876.

(20.) Difference between curve of pressure and line of resistance. Illustrated by Dubois under the names "pressure line" and "support line" in his "Graphical Statics," Art. 176 and Fig. 102.

(21.) *Unsymmetrical loading.* See Collignon's general theorem, "Résistance des Matériaux," Art. 225

Examples of Arches.

Morandière, "Traite de la Construction des Ponts et Viaducs." Masonry Bridges, pages 1 to 508; and plates 34 to 136, being 12″ x 18″ engravings. Paris, 1876.